

# Slow Roll Inflation in Non-Minimally Coupled Theories: Hyperextended Gravity Approach

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## Abstract

The slow roll approximation is studied for cosmological models in Hyperextended Scalar-Tensor Theories of Gravity. A procedure to obtain slow roll solutions in non-minimally coupled gravity is outlined and some examples are provided. An integral condition over the functional form of the non-minimal coupling is imposed in order to obtain intermediate inflationary behavior.

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## I. INTRODUCTION

In this work we analyze the slow roll approximation in cosmological models of non-minimally coupled theories of gravity (NMC) starting from the complete two free functions lagrangian density of Hyperextended Scalar-Tensor Theories (HSTG) [1]. These HSTG are different from both, generalized Brans-Dicke (BD) cases and NMC cases, when there are non-invertible functions of the field involved in the lagrangian. This give rise to the representation of a singularity in the field transformation which allows the passage from a NMC lagrangian to a BD one [2]. HSTG may reduce itself either to BD or to NMC whenever convenient. In past work, exact cosmological solutions were obtained for these models in the case of vacuum, radiation and stiff matter universes [1]. Concerning inflation, exact and approximate solutions were found for particular scalar-tensor theories [3]. The slow roll approximation [4] were analyzed by Garcia Bellido, Linde and Linde in the case of BD gravity [5] and later by Barrow for generalized BD gravity with power law couplings [6]. Here, we are going to present the slow roll approximation in the context of HSTG. We work out, following the method used by Barrow, examples for non-minimally coupled gravity. We impose an integral vinculum over the form of the non-minimal coupling in order to have intermediate inflation behavior.

## II. HYPEREXTENDED SCALAR-TENSOR THEORIES AND THE SLOW ROLL APPROXIMATION

The hyperextended scalar-tensor field equations are derived from the action:

$$S = \int d\Omega \sqrt{-g} \left[ 16\pi L_M + \frac{K(\phi)}{2} \phi_{,\mu} \phi^{,\mu} + G(\phi)^{-1} R \right] \quad (1)$$

From now on, we shall call  $K(\phi) = \frac{-2\omega(\phi)}{\phi}$  to facilitate the comparison with the BD cases. This must be understood only as a change in the names of the functions. Taking variational derivatives from (1) with respect to the dynamical variables  $g^{\mu\nu}$  and  $\phi$  we get the field equations. We are interested in an isotropical and homogeneous universe. We shall have

a metric defined by a Friedmann-Robertson-walker one with zero curvature and a stress-energy tensor for matter given by the deduced from a lagrangian density of another scalar field  $\sigma$  with self interaction potential  $V(\sigma)$ . Finally, these equations become:

$$H^2 - H \frac{1}{G} \frac{dG}{d\phi} \dot{\phi} - \frac{\omega}{6} \frac{\dot{\phi}^2}{\phi} G = \frac{8\pi}{3} G \left[ \frac{1}{2} \dot{\sigma}^2 + V(\sigma) \right] \quad (2)$$

$$\dot{\phi}^2 \Gamma + \square \phi \left[ \frac{2\omega}{\phi} + \frac{3}{G^3} \left( \frac{dG}{d\phi} \right)^2 \right] = -\frac{1}{G} \frac{dG}{d\phi} 8\pi \left[ 4V(\sigma) - \dot{\sigma}^2 \right] \quad (3)$$

$$\ddot{\sigma} + 3H\dot{\sigma} + V'(\sigma) = 0 \quad (4)$$

where we have defined

$$\Gamma = \left[ \frac{1}{\phi} \frac{d\omega}{d\phi} - \frac{\omega}{\phi^2} - \frac{1}{G} \frac{dG}{d\phi} \frac{\omega}{\phi} - \frac{6}{G^4} \left( \frac{dG}{d\phi} \right)^3 + \frac{3}{G^3} \frac{dG}{d\phi} \frac{d^2 G}{d\phi^2} \right] \quad (5)$$

and  $H$  as usual. These equations reduce to those of GR when  $\phi$  equals to a constant and to those of BD when  $G$  equals  $1/\phi$ .

In order to examine the forms of inflation that can arise in these kind of models, we are going to assume the slow roll approximation given by:

$$\ddot{\sigma} \ll H\dot{\sigma} \quad (6)$$

$$\frac{1}{2} \dot{\sigma}^2 \ll V(\sigma) \quad (7)$$

$$\ddot{\phi} \ll H\dot{\phi} \ll H^2 \phi \quad (8)$$

The last condition requires a scalar field which evolves slowly with respect to the expansion of the universe and was studied and applied in the context of both, BD gravity [5] and generalized BD gravity [6]. Condition (8) may be used to fix a similar one, suitable to be applied in the more general case. It is possible to see that, for any function  $f$  positive defined and with a convergent Taylor serie, condition (8) may be transformed to

$$\ddot{f}(\phi) \ll H\dot{f}(\phi) \ll H^2 f(\phi) \quad (9)$$

in any case in which  $\sim \exists n/n \in \mathbf{N} \wedge n\dot{\phi} \simeq H\phi$ . In particular, the choice  $f = G^{-1}$  makes possible to fix useful simplifications on the field equations (2-4). Finally, the slow roll field equations become:

$$3H\dot{\sigma} \simeq -V'(\sigma) \quad (10)$$

$$3H^2 \simeq \frac{\omega}{2} \frac{\dot{\phi}^2}{\phi} G + \frac{8\pi}{3} GV(\sigma) \quad (11)$$

$$\dot{\phi}^2 \Gamma + 3H\dot{\phi} \left[ \frac{2\omega}{\phi} + \frac{3}{G^3} \left( \frac{dG}{d\phi} \right)^2 \right] \simeq -\frac{1}{G} \frac{dG}{d\phi} 32\pi V(\sigma) \quad (12)$$

The leading order terms referred in the slow roll equations also reduce to the BD ones when  $G = 1/\phi$  as expected (see reference [6] for comparison). From here, and in order to proceed further, it is necessary to define both,  $\omega(\phi)$  and  $G(\phi)$  together with the form of the potential  $V(\sigma)$  for the inflaton.

In [6], a complete fixing of the problem was obtained by defining the functional form of the coupling, in that case a power law. We propose now, using the freedom given by having two generic functions instead of one, to sketch how general analytical solutions for the slow roll equations for non-minimally coupled theories may be obtained. From now on, we are going to use equalities even when we only have approximate expressions.

### III. SLOW ROLL SOLUTIONS OF NON-MINIMALLY COUPLED GRAVITY

The NMC theories of gravity were obtained in the previous formalism defining  $\omega = -\phi/2$ . We shall impose that, in order to recover the Einstein's regime at large times and to be consistent with solar system tests of gravity,  $\frac{(G^{-1})^4}{(G^{-1})^3} \rightarrow 0$  when  $t \rightarrow \infty$  [7]. This condition is satisfied when  $G^{-1}$  tends to a constant without asymptotic variations in the first derivative. This *matching approximation* carries the factor  $\Gamma$  in the previous equations to  $\Gamma_a$ , given by:

$$\Gamma_a = \left[ \frac{1}{\phi} \frac{d\omega}{d\phi} - \frac{\omega}{\phi^2} - \frac{1}{G} \frac{dG}{d\phi} \frac{\omega}{\phi} \right] \quad (13)$$

Finally, the NMC slow roll equations may be written as

$$3H\dot{\sigma} = -V'(\sigma) \quad (14)$$

$$3H^2 = -\frac{1}{4}\dot{\phi}^2 G + 8\pi G V(\sigma) \quad (15)$$

$$\dot{\phi}^2 \frac{1}{G} \frac{dG}{d\phi} + 3H\dot{\phi} \left[ -1 + \frac{3}{G^3} \left( \frac{dG}{d\phi} \right)^2 \right] = -\frac{1}{G} \frac{dG}{d\phi} 32\pi V(\sigma) \quad (16)$$

A further application of the slow roll condition on (16) allows to write it as

$$3H\dot{\phi} \left[ -1 + \frac{3}{G^3} \left( \frac{dG}{d\phi} \right)^2 \right] = -\frac{1}{G} \frac{dG}{d\phi} 32\pi V(\sigma) \quad (17)$$

Using (15) in (17) and supposing that  $\phi^2 G$  does not diverge we obtain an additional product of the slow roll equations:

$$3H\dot{\phi} \left[ -1 + \frac{3}{G^3} \left( \frac{dG}{d\phi} \right)^2 \right] = -\frac{1}{G} \frac{dG}{d\phi} 12H^2 G^{-1} \quad (18)$$

Note that the second term on the left behaves as  $-9\frac{1}{G}\frac{dG}{d\phi}H(G^{-1})$  while that on the right does as  $-12\frac{1}{G}\frac{dG}{d\phi}H^2G^{-1}$  and so we are allowed to have another simplification

$$3H\dot{\phi} = \frac{1}{G^2} \frac{dG}{d\phi} 12H^2 \quad (19)$$

This last equation may be readily integrated as

$$\int d\phi \frac{-1}{\frac{dG^{-1}}{d\phi}} = 4 \ln \left( \frac{a}{a_0} \right) \quad (20)$$

with  $a_0$  a constant of integration; thus giving  $\phi = \phi(a)$  for every selection which makes invertible the result of the integral and  $a = a(\phi)$  always.

To find  $\phi = \phi(t)$  it is necessary to specify the form of the potential. Let us consider some common cases.

**A.**  $V(\sigma) = V_0 = \text{constant}$

The case of constant potential allows complete integration. From the previous paragraph we have

$$3H^2G^{-1} = 8\pi V_0 \quad (21)$$

Replacing the value of  $H$  in (19) we obtain

$$\int d\phi \frac{-1}{G^{1/2} \frac{dG^{-1}}{d\phi}} = 4 \left( \frac{8\pi V_0}{3} \right)^{\frac{1}{2}} (t - t_0) \quad (22)$$

with  $t_0$  a constant of integration. Defining the gravitational theory by giving the form of  $G$ , (22) gives  $t = t(\phi)$  and, in the cases in which inversion is possible,  $\phi = \phi(t)$ . In such cases, it will be also possible then, to replace in (20) to have  $a = a(t)$ .

We see that the procedure followed here continues the line of that presented by Barrow [6] in the case of given coupling of a BD theory; and as that, this enable to see the form of inflation (if any) in a particular gravity theory by integrating and inverting two differential equations.

**B.**  $V(\sigma) = V_0 \exp(-\lambda\sigma)$ ;  $V_0, \lambda = \text{constants}$

With this exponential form for the inflaton potential the slow roll equations become

$$3H\dot{\sigma} = \lambda V_0 \exp(\lambda\sigma) \quad (23)$$

$$3H^2G^{-1} = 8\pi V_0 \exp(-\lambda\sigma) \quad (24)$$

$$3H\dot{\phi} \left[ -1 + \frac{3}{G^3} \left( \frac{dG}{d\phi} \right)^2 \right] = -\frac{1}{G} \frac{dG}{d\phi} 32\pi V_0 \exp(\lambda\sigma) \quad (25)$$

Defining as in [6] a new time coordinate as

$$t = \int 3H d\eta \quad (26)$$

the equations are integrable again:

$$\sigma(\eta) = \frac{1}{\lambda} \ln [\lambda^2 V_0 (\eta + \eta_0)] \quad (27)$$

$$\int d\phi \frac{-1 + \frac{3}{G^3} \left(\frac{dG}{d\phi}\right)^2}{G \frac{dG^{-1}}{d\phi}} = 32\pi \lambda^{-2} \ln [\eta + \eta_0] \quad (28)$$

Defining  $G$  and inverting (28) to get  $\phi$  as a function of  $\eta$  one can use the slow roll equation (24) to finally obtain the behavior of  $H$ .

$$\mathbf{C.} \ V(\sigma) = V_0 \sigma^{2r}; \ V_0, r = \text{constant} \ \mathbf{and} \ r \neq 1$$

With the same time variable, we have now:

$$\sigma' = -2r V_0 \sigma^{2r-1} \quad (29)$$

$$3H^2 G^{-1} = 8\pi V_0 \sigma^{2r} \quad (30)$$

$$\phi' \left[ -1 + \frac{3}{G^3} \left(\frac{dG}{d\phi}\right)^2 \right] = -\frac{1}{G} \frac{dG}{d\phi} 32\pi V_0 \sigma^{2r} \quad (31)$$

and the solutions are:

$$\sigma(\eta) = [4r(1-r)V_0(\eta_0 - \eta)]^{\frac{1}{2(1-r)}} \quad (32)$$

$$\int d\phi \frac{-1 + \frac{3}{G^3} \left(\frac{dG}{d\phi}\right)^2}{G \frac{dG^{-1}}{d\phi}} = 32\pi V_0 [4r(1-r)V_0]^{\frac{r}{1-r}} (\eta_0 - \eta)^{\frac{1}{(1-r)}} \quad (33)$$

with  $\eta_0$  a constant of integration.

#### IV. SLOW ROLL SOLUTIONS: EXAMPLES

Let us first show how the formalism works for integrable and invertible examples which do not require numerical recipes and admit comparison with previous works.

**A.**  $G^{-1} = \phi^n$ ,  $n \neq 2, 4$  **and**  $V = V_0 = \text{constant}$

We are dealing with the lagrangian density given by

$$L = \phi^n R + \frac{1}{2} \phi_{,\mu} \phi^{,\mu} + 16\pi L_m \quad (34)$$

Equation (20) may be integrated to give

$$\phi(a)^{-n+2} = -4n(2-n) \ln(a/a_0) \quad (35)$$

Equation (22) may also be integrated:

$$\phi(t)^{-n/2+2} = -4n \left( -\frac{n}{2} + 2 \right) \left( \frac{8\pi V_0}{3} \right)^{\frac{1}{2}} t \quad (36)$$

and so

$$\phi(t) \propto t^{\frac{2}{-n+4}} \quad (37)$$

This last relationship, used in (35) gives

$$\left( \frac{a(t)}{a_0} \right) \propto \exp \left[ t^2 \left( \frac{-n+2}{-n+4} \right) \right] \quad (38)$$

When  $n$  grows, the behavior of the scale factor tends to  $\left( \frac{a(t)}{a_0} \right) \propto \exp[t^2]$ . We have to consider now if these solutions are consistent with the slow roll approximations. We see that  $\frac{\dot{\phi}}{H\phi} \ll 1$  for all  $n$  greater than 4 and lower than 2 and that the approximation fails in the interval  $(2, 4)$ . The slow roll solutions found have the form of those of intermediate inflation, in which the scale factor expands more slowly than for De Sitter case but faster than for power law inflation [3]. The consistency of the  $\Gamma$  to  $\Gamma_a$  passage may be also tested, we find that  $\frac{dG^{-1}}{d\phi} \propto t^{-2\frac{2-n}{4-n}}$  and so, it is a decreasing quantity with  $t$ .

**B.**  $G^{-1} = \phi^2$ ,  $V = V_0 = \text{constant}$

The lagrangian density given by

$$L = \phi^2 R + \frac{1}{2} \phi_{,\mu} \phi^{,\mu} + 16\pi L_m \quad (39)$$



was mainly studied by the Naples group in relation with scalar potentials and inflation [8]. Using equation (20) we get

$$\phi(a) \propto \left(\frac{a}{a_0}\right)^{-8} \quad (40)$$

and with equation (22)

$$\phi(t) = ct \quad (41)$$

with  $c$  a constant. It may be verified with these solutions that both, the universe is not in expansion and the slow roll condition is not satisfied; and so, they must be discarded.

$$\mathbf{C.} \ G^{-1} = \phi^4, \ V = V_0 = \text{constant}$$

Equation (20) gives in this case

$$\frac{a}{a_0} \propto \exp[\phi^{-2}] \quad (42)$$

while (22),

$$\phi \propto \exp[-t] \quad (43)$$

Thus, the behavior of the scale factor with time is given by

$$a \propto \exp[\exp[2t]] \quad (44)$$

Here, the scale factor appears to exhibit an extreme form of inflation. It may be proved that  $|\frac{\dot{\phi}}{H\phi}| \ll 1$  and  $\frac{d^2 G^{-1}}{d\phi^2} \rightarrow 0$  when  $t$  is large enough. The evolution of  $\sigma(\eta)$  can be found from equation (4) in each of the cases previously analized since  $\dot{\sigma} \propto a^{-3}$ .

#### **D. Spaning of Intermediate Inflation Solutions**

We are willing now to study in a completely general form which kind of couplings may allow intermediate inflation behavior. For simplicity we shall use the case of constant potential.

In our study, we shall have two important equations: (20), which is potential independent, and (22). From the integration of (20), it may immediately be seen that, in order to have a scale factor evolving as

$$a(t) \propto \exp[t^m] \quad (45)$$

it is necessary to have  $t^m = \int d\phi \frac{-1}{\frac{dG}{d\phi}}$ , that is:

$$\left[ \int d\phi \frac{-1}{G^{1/2} \frac{dG}{d\phi}} \right]^m = \int d\phi \frac{-1}{\frac{dG}{d\phi}} \quad (46)$$

Deriving with respect to  $\phi$  we get:

$$m \int d\phi \frac{G^2}{\frac{dG}{d\phi}} = G^{\frac{1}{2}} \int d\phi \frac{G^2}{\frac{dG}{d\phi} G^{\frac{1}{2}}} \quad (47)$$

which is an integral vinculum over the functional form of  $G(\phi)$ . Although highly non-linear, equation (47) is very suggestive. It may be proved that  $G(\phi) = \phi^\alpha$  with  $m = \frac{2(2+\alpha)}{4+\alpha}$  is one of its solutions, as expected, since we find intermediate inflation behavior for it in the previous section. This case was already studied concerning the fulfillment of the slow roll conditions. Also the choice  $G(\phi) = \alpha \exp[\alpha\phi]$ , with  $\alpha$  a constant, is a solution of (47). For such choice, using the previously derived method we obtain:

$$a \propto \exp[t^2] \quad (48)$$

and

$$\phi \propto \frac{-2}{\alpha} \ln[t] \quad (49)$$

It is worth recalling that the behavior of the scale factor in (48) coincides with that of  $G = \phi^\alpha$  if  $\alpha$  is large enough. The exponential form of  $G$  is such that the slow roll and the  $\Gamma \rightarrow \Gamma_a$  conditions are fulfilled.

## V. DISCUSSION

Starting from the hyperextended approach of scalar-tensor theories, we have formulated the slow roll approximation in non-minimally coupled gravity. The solutions were given in

the form of two integrals that may be solved analytically in some cases and numerically in all cases. These integrals give the behavior of  $\phi$  as function of  $a$  and  $t$ . Properly inversion (again analytically or numerically) gives so,  $a$  and  $\phi$  as functions of time. We studied constant, power law and exponential forms for the inflaton potential. Some examples that show intermediate inflation were presented for power law non-minimally couplings in agreement with the results of power-law couplings in BD theories. An integral vinculum for the functional form of  $G$  was established in order to develop intermediate inflation behavior and some examples were provided.

## VI. ACKNOWLEDGMENTS

This work was partially supported by CONICET. Valuable conversations with H. Vucetich are acknowledged.

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